Local convexity control of a bivariate rational interpolation function

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Local sharp control of surface model is the fundamental issues of surface modeling in the field of CAD/CG. A bivariate rational interpolation function based on function values and partial derivatives was constructed. The local convexity control method of the interpolating surfaces is employed to control the shape of surfaces, and the sufficient and necessary conditions for the interpolating surfaces to be convex are derived. The convexity of the interpolating surface can be changed locally by selecting suitable parameters under the condition that the interpolation data are not changed. Also, the numerical examples are presented to show the performance of the method.

Introduction

The construction method of the surface and the mathematical description, shape control of them is a key issue in computer-aided design. In order to meet the needs of the ever-increasing model complexity and to incorporate manufacturing requirements, shape control becomes an ever more important task in constructing curves and surfaces. There are some methods for preserving positivity or preserving convexity in the design of surfaces [1-4], but there are few modification methods to control the shape of the interpolating surface [6,9-12].

In recent years, the univariate rational spline interpolation with parameters has been constructed [8-12]. Those kinds of interpolation spline not only have simple mathematical representation, they can be used for the modification of local curves by selecting suitable parameters. There are few such bivariate interpolating splines which have simple and explicit mathematical representation and can be modified under the conditions that the interpolation data are not changed. Motivated by the univariate rational spline interpolation, the bivariate rational interpolation with parameters has been constructed[12]. Since there are parameters in the interpolation function, the interpolating surface varies as the parameters change. More importantly, it is a convexity preserving interpolation of surface. This paper will deal with the convexity control method to modify the convexity of the interpolating surface based on the definition of the surface's convexity defined by the Gauss curvature.

The paper is arranged as follows. In Section 1, the new bivariate rational spline based on function values and partial derivatives will be restated. Section 2 discusses the bases function of this bivariate interpolation. Section 3 deals with the convexity control of interpolating surfaces. In Section 4, examples are given to show the performance of the method.

1. Interpolation function

Let $\Omega:[a,b;c,d]$ be the plane region, and $\{(x_i,y_j,f_{i,j},\frac{\partial f_{i,j}}{\partial x},\frac{\partial f_{i,j}}{\partial y}),\ i=1,\ 2,\ldots,n;j=1,2,\ldots,m\}$ be a given set of data points. Let $h_i=x_{i+1}-x_i$, $l_j=y_{j+1}-y_j$, and for (x,y) any point $(x,y)\in[x_i,x_{i+1};y_j,y_{j+1}]$ in the (x,y)-plane, and let $\theta=\frac{x-x_i}{h_i}$ and $\eta=\frac{y-y_j}{l_j}$. First, for each $y=y_j$, $j=1,2,\ldots m$, construct the x-direct interpolating curve $P^*_{i,j}(x)$ in $[x_i,x_{i+1}]$, this is given by

$$P^*_{i,j}(x)=\frac{p^*_{i,j}(x)}{q^*_{i,j}(x)},i=1,2,\cdots n-1, \qquad (1)$$

For each pair (i,j), let $\beta_{i,j}>0$, define the bivariate interpolating function $P_{i,j}(x,y)$ on $[x_i,x_{i+1};y_j,y_{j+1}]$ as follows

$$P_{i,j}(x,y)=\frac{p_{i,j}(x,y)}{q_{i,j}(y)},i=1,2,\cdots n-1;j=1,2,\cdots m-1, \qquad (2)$$

The term $P_{i,j}(x,y)$ is called the bivariate rational interpolation based on function values and partial derivative values which satisfies

$$P_{i,j}(x_r,y_s)=f(x_r,y_s),\frac{\partial P_{i,j}(x_r,y_s)}{\partial x}=\frac{\partial f_{r,s}}{\partial x},\frac{\partial P_{i,j}(x_r,y_s)}{\partial y}=\frac{\partial f_{r,s}}{\partial y}. \qquad (3)$$

It was proved that when the parameters $\beta_{i,j}$ is constant, the interpolating function $P_{i,j}(x,y)$ must be C^1 continuous in the whole interpolating region.

2. Interpolation Basis function

From (1), (2) and (3) the interpolating function $P_{i,j}(x,y)$ defined by (2) can be written as the follows

$$P_{i,j}(x,y)=\sum_{r=i}^{i+1}\sum_{s=j}^{j+1}[a_{r,s}(\theta,\eta)f_{r,s}+b_{r,s}(\theta,\eta)h_i\frac{\partial f_{r,s}}{\partial x}+c_{r,s}(\theta,\eta)l_j\frac{\partial f_{r,s}}{\partial y}], \quad (4)$$

The terms $a_{r,s}(\theta,\eta)$, $b_{r,s}(\theta,\eta)$, $c_{r,s}(\theta,\eta)$ are called the bases of the interpolation, and (4) is its more explicit form of the interpolation.

2.1. Basic derivative properties of the interpolating function

For the bivariate rational interpolating function $P_{i,j}(x,y)$, consider the first-order derivatives.

$$\frac{\partial\omega_1(\theta,\alpha_i)}{\partial x}=\frac{\partial\omega_2(\theta,\alpha_i)}{\partial x},\frac{\partial\omega_1(\eta,\beta_j)}{\partial y}=\frac{\partial\omega_2(\mu,\beta_i)}{\partial y},\frac{\partial a_{i+r,j+s}(\theta,\eta)}{\partial x}=\omega_{1+s}(\eta,$$

$$\beta_j)\frac{\partial\omega_{1+r}(\theta,\alpha_i)}{\partial x},\frac{\partial b_{i+r,j+s}(\theta,\eta)}{\partial y}=\omega_{3+s}(\theta,\alpha_i)\frac{\partial\omega_{1+r}(\eta,\beta_j)}{\partial y},(r,s=0,1)$$

Property 1. If $P_{i,j}(x,y)$ is the interpolating function in $[x_i,x_{i+1};y_j,y_{j+1}]$ defined by (2), no matter what positive numbers the parameters α_i, β_j take, the unity property holds.

3. Conditions for the interpolating surface to be convex

To study the convexity control of the interpolating surfaces, the conditions for the interpolating surfaces to be convex at a point need to be derived. Before this, gives the definition of the convex surface.

Definition 1. If Gauss curvature at each point of a surface is positive, the surface is called the convex surface[13].

Lemma 1. For the given interpolation function $P_{i,j}(x,y)$, the sufficient and nec-essary conditions for a surface $P_{i,j}(x,y)$ to be convex in $[x_i,x_{i+1};y_j,y_{j+1}]$ is that for any $(x,y)\in[x_i,x_{i+1};y_j,y_{j+1}]$.

$$\frac{\partial^2P_{i,j}(x,y)}{\partial x^2}\frac{\partial^2P_{i,j}(x,y)}{\partial y^2}-(\frac{\partial^2P_{i,j}(x,y)}{\partial x\partial y})^2>0. \quad (5)$$

From Property 1, the following Eqs. can be derived.

$$\frac{\partial^2P_{i,j}(x,y)}{\partial x^2}\frac{\partial^2P_{i,j}(x,y)}{\partial y^2}-(\frac{\partial^2P_{i,j}(x,y)}{\partial x\partial y})^2=\frac{1}{h_i^2l_j^2}\frac{1}{(q_{i,j}^*(x))^4(q_{i,j}(y))^4}E(\theta$$

$$,\eta;\alpha_i,\beta_j)\text{, where } E(\theta,\eta;\alpha_i,\beta_j)=4(X_1Y_1+X_2(\alpha_iY_2(b_1,b_3)+Y_2(b_2,b_4)$$

$$-\alpha_i^2))\cdot(X_1'Y_1'+X_2'(c_1,c_2)Y_2'+X_2'(c_3,c_4)Y_3')-(X_1^*Y_1^*-X_2^*Y_2^*)^2 \quad (6)$$

then (5) is equivalent to $E(\theta,\eta;\alpha_i,\beta_j)>0$ (7)

Theorem 1. For the given interpolation data $f_{i,j},\frac{\partial f_{i,j}}{\partial x},\frac{\partial f_{i,j}}{\partial y}$, $r=i,i+1;\ s=j,j+1$, $P_{i,j}(x,y)$, the sufficient and necessary conditions for surface $P_{i,j}(x,y)$ to be convex in a point $(x,y)\in[x_i,x_{i+1};y_j\ \ ,y_{j+1}]$ is that for the corresponding local coordinate (θ,η) of point (x,y) and the parameters α_i,β_j, inequality (7) holds.

4. Numerical examples

Example 1. Let $\Omega:[0,1;0,1]$ be the plane region, and the interpolation data are given in Table 1.

Table 1

The interpolating data

(x_i,y_j)	$f_{i,j}$	$\frac{\partial f_{i,j}}{\partial x}$	$\frac{\partial f_{i,j}}{\partial y}$
(0, 0)	3.0	1.5	-0.5
(0, 1)	6.0	1.0	2.0
(1, 0)	2.0	-2.8	2.0
(1, 1)	5.0	-1.6	-0.2

Table 2

The interpolating data

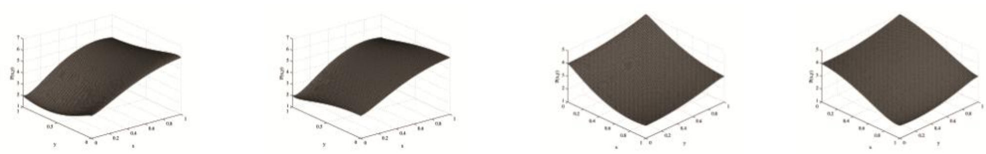
(x_i,y_j)	$f_{i,j}$	$\frac{\partial f_{i,j}}{\partial x}$	$\frac{\partial f_{i,j}}{\partial y}$
(0, 0)	4.0	-0.5	-0.4
(0, 1)	2.0	1.0	-1.5
(1, 0)	5.0	-0.8	2.0
(1, 1)	3.0	-0.2	-0.3

Now, let $\alpha_i = \mathbf{0.15}$; $\beta_j = \mathbf{0.08}$, Figure 1(a) shows the graph of the surface $\boldsymbol{P_1(x,y)}$. $\theta = \eta = \mathbf{0.5}$, it is easy to compute $\boldsymbol{E(\theta,\eta;\alpha_i,\ \beta_j) < 0}$. Therefore, $\boldsymbol{P_1(x,y)}$ is concave at the point (0.5,0.5). Let $\alpha_i = \mathbf{0.15}$; $\beta_j = \mathbf{8.0}$, in this case, $\boldsymbol{E(\theta,\eta;\alpha_i,\beta_j) > 0}$. Thus, the interpola-ting surface must be convex at the point (0.5,0.5). Figure 1 (b)shows the graph of the surface $\boldsymbol{P_2(x,y)}$.

Example 2. Let $\boldsymbol{\Omega : [0,1;0,1]}$ be the plane region, and the interpolation data are given in Table 2.
We take $\alpha_i = \mathbf{3.2}$; $\beta_j = \mathbf{7.5}$, and denote the interpolation function by $\boldsymbol{P_5(x,y)}$. Taking $\alpha_i = \mathbf{0.05}$, $\beta_j = \mathbf{10.2}$, denote the interpolation function by $\boldsymbol{P_6(x,y)}$. For the point (0.5,0.5), it is easy to compute that $\boldsymbol{E(\theta,\eta;\alpha_i,\beta_j) < 0}$ and $\boldsymbol{E(\theta,\eta;\alpha_i,\beta_j) > 0}$,respectively. Therefore the surface $\boldsymbol{P_5(x,y)}$ and $\boldsymbol{P_6(x,y)}$ are conc-ave at the point (0.5,0.5). Figure 1(c)(d) shows the graph of the surface.

5. Concluding remarks

The convex discriminant function and the convex discriminant inequality of the interpolating surface have been derived in this paper, based on the convex surface definition given by the Gauss curvature. In order to determine that the surface is convex or not at a point $\boldsymbol{(x,y)}$, we need to test that the inequality (7) holds or not at this point. Also, in order to modify the convexity, suitable positive parameters α_i and β_j can be chosen in a simple way involving the computation of the convex discriminant function $\boldsymbol{E(\theta,\eta;\alpha_i,\beta_j)}$. According to the $\boldsymbol{E(\theta,\eta;\alpha_i,\ \beta_j)}$ value, the degree of convexity can be judged approx-imately.



(a) (b) (c) (d)

Fig. 1 This is a single-column figure

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